

N63 22 141

Code 1

(NASA CR-51319)

JPL-TR 32469

Technical Report No. 32-469

*A Table of Integrals Involving Powers,
Exponentials, Logarithms, and the
Exponential Integral*

Murray Geller

Aug. 1, 1963

JPL

JPL-TR 32469

JPL-TR 32469

(NASA CR-51319)

jpl

4742683

JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

August 1, 1963

OTS PRICE

XEROX

\$

~~XXXXXXXXXX~~

MICROFILM

\$

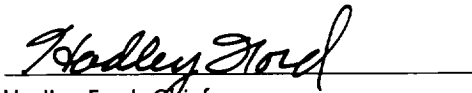
~~XXXXXXXXXX~~

CASE FILE COPY

Technical Report No. 32-469

*A Table of Integrals Involving Powers,
Exponentials, Logarithms, and the
Exponential Integral*

Murray Geller

A handwritten signature in cursive script, reading "Hadley Ford", is written over a horizontal line.

Hadley Ford, Chief
Chemistry Section

JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

August 1, 1963

Copyright © 1963
Jet Propulsion Laboratory
California Institute of Technology

Prepared Under Contract No. NAS 7-100
National Aeronautics and Space Administration

CONTENTS

Introduction	1
I. Integrals of the Type $\int x^p e^{-ax} dx$	2
A. $A(p, \alpha) = \int_0^{\infty} x^p e^{-ax} dx$	2
B. $A_1(p, \alpha, \rho) = \int_0^{\rho} x^p e^{-ax} dx$	2
C. $A_2(p, \alpha, \rho) = \int_{\rho}^{\infty} x^p e^{-ax} dx$	2
II. Integrals of the Type $\int x^p e^{-ax} \ln x dx$	3
A. $B(p, \alpha) = \int_0^{\infty} x^p e^{-ax} \ln x dx$	3
B. $B_1(p, \alpha, \rho) = \int_0^{\rho} x^p e^{-ax} \ln x dx$	4
C. $B_2(p, \alpha, \rho) = \int_{\rho}^{\infty} x^p e^{-ax} \ln x dx$	4
III. Integrals of the Type $\int x^p e^{-ax} \ln^2 x dx$	5
A. $C(p, \alpha) = \int_0^{\infty} x^p e^{-ax} \ln^2 x dx$	5
B. $C_1(p, \alpha, \rho) = \int_0^{\rho} x^p e^{-ax} \ln^2 x dx$	6
C. $C_2(p, \alpha, \rho) = \int_{\rho}^{\infty} x^p e^{-ax} \ln^2 x dx$	6
IV. Integrals of the Type $\int x^p e^{-ax} \ln^3 x dx$	7
A. $D(p, \alpha) = \int_0^{\infty} x^p e^{-ax} \ln^3 x dx$	7
B. $D_1(p, \alpha, \rho) = \int_0^{\rho} x^p e^{-ax} \ln^3 x dx$	8
C. $D_2(p, \alpha, \rho) = \int_{\rho}^{\infty} x^p e^{-ax} \ln^3 x dx$	8
V. Miscellaneous Related Integrals	9
A. $F(n, \alpha, p) = \int_0^{\infty} x^p e^{-ax} \ln^n x dx$	9
B. $F_1(n, \alpha, p, \rho) = \int_0^{\rho} x^p e^{-ax} \ln^n x dx$	9
C. $\int_{\rho}^{\infty} e^{-ax} \ln^n x dx$	9

CONTENTS (Cont'd)

VI. Integrals of the Type $\int x^p [-E_i(-\beta x)] e^{-\alpha x} dx$	10
A. $G(p, \alpha, \beta) = \int_0^\infty x^p e^{-\alpha x} [-E_i(-\beta x)] dx$	10
B. $G_1(p, \alpha, \beta, \rho) = \int_0^\rho x^p e^{-\alpha x} [-E_i(-\beta x)] dx$	11
C. $G_2(p, \alpha, \beta, \rho) = \int_\rho^\infty x^p e^{-\alpha x} [-E_i(-\beta x)] dx$	12
VII. Integrals of the Type $\int x^p e^{-\alpha x} \ln x [-E_i(-\beta x)] dx$	12
A. $H(p, \alpha, \beta) = \int_0^\infty x^p e^{-\alpha x} \ln x [-E_i(-\beta x)] dx$	12
B. $H_1(p, \alpha, \beta, \rho) = \int_0^\rho x^p e^{-\alpha x} \ln x [-E_i(-\beta x)] dx$	13
C. $H_2(p, \alpha, \beta, \rho) = \int_\rho^\infty x^p e^{-\alpha x} \ln x [-E_i(-\beta x)] dx$	15
VIII. Integrals of the Type $\int x^p e^{-\alpha x} \ln^2 x [-E_i(-\beta x)] dx$	16
A. $I(p, \alpha, \beta) = \int_0^\infty x^p e^{-\alpha x} \ln^2 x [-E_i(-\beta x)] dx$	16
B. $I_1(p, \alpha, \beta, \rho) = \int_0^\rho x^p e^{-\alpha x} \ln^2 x [-E_i(-\beta x)] dx$	18
C. $I_2(p, \alpha, \beta, \rho) = \int_\rho^\infty x^p e^{-\alpha x} \ln^2 x [-E_i(-\beta x)] dx$	18
IX. Integrals of the Type $\int x^p e^{-\alpha x} \ln^3 x [-E_i(-\beta x)] dx$	19
A. $J(p, \alpha, \beta) = \int_0^\infty x^p e^{-\alpha x} \ln^3 x [-E_i(-\beta x)] dx$	19
B. $J_1(p, \alpha, \beta, \rho) = \int_0^\rho x^p e^{-\alpha x} \ln^3 x [-E_i(-\beta x)] dx$	19
C. $J_2(p, \alpha, \beta, \rho) = \int_\rho^\infty x^p e^{-\alpha x} \ln^3 x [-E_i(-\beta x)] dx$	19
X. Miscellaneous Related Integrals	20
A. $K(n, \alpha, \beta, \rho) = \int_0^\infty x^p e^{-\alpha x} \ln^n x [-E_i(-\beta x)] dx$	20
B. $\alpha \int_\rho^\infty e^{-\alpha x} \ln^n x [-E_i(-\beta x)] dx$	20
C. $\int_\rho^\infty \frac{e^{-\alpha x} [-E_i(-\beta x)]}{x} dx$	20
D. $K_1(n, \alpha, \beta, p, \rho) = \int_0^\rho x^p e^{-\alpha x} \ln^n x [-E_i(-\beta x)] dx$	20
Nomenclature	21
References	22

PREFACE

This project was conducted under a research grant from the National Science Foundation at the Department of Chemistry, Johns Hopkins University, Baltimore, Maryland.

ABSTRACT

22/41

The general integrals $\int_a^b x^p e^{-ax} (\ln x)^n dx$ and $\int_a^b x^p e^{-ax} (\ln x)^n [-Ei(-\beta x)] dx$ are investigated, where n is an integer, a and β are real positive numbers, and p is a number greater than -1 . Many special cases are obtained, and the results are tabulated in a logical order. Where possible the integrals are expressed in closed form, and several cases are expressed in series expansions.

INTRODUCTION

The following table is a collection of some frequently occurring integrals in quantum mechanics among other applications involving powers, exponentials, logarithms and exponential integrals. Where possible the integrals are expressed in closed form. Also included are several integrals which are expressed in series expansions. It is hoped that these expansions may be useful for many purposes.

The first four sections of the tables deal with powers, exponentials, and logarithms to the zero power, first power, second power, and third power respectively. The fifth section involves some generalizations of the logarithm to the n th power and some miscellaneous results. The same arrangement applies to the sixth through tenth sections except that the exponential integral is included. Within each section, the first subsection is the integral from 0 to ∞ , the second from 0 to ρ , and the third from

ρ to ∞ . Lastly, within each subsection, the general result is obtained for x^p , then examples are given for $p=0,1,2,3$ and for integer n and finally miscellaneous interesting examples, notes and definitions. Throughout the tables p represents any number greater than -1 and n,i,k,l represent integers.

The notation followed consistently is that of Erdelyi, Magnus, Oberhettinger, and Tricomi (Ref. 1). Several of the integrals presented here can be found in the tables of Bierens de Haan (Ref. 2), Hofrieter and Grobner (Ref. 3), Ryshik and Gradstein (Ref. 4), and LeCaine (Ref. 5) and the books of Erdelyi *et al.* (Ref. 1), and Nielsen (Ref. 6). Other references containing pertinent integrals are Levenson (Ref. 7), who discusses the integral denoted $F(n,1,0)$ in Sec. V, Eq. 1, and Busbridge (Ref. 8) and Kourganoff (Ref. 9), who discuss the generalization of $G(p,\alpha,\beta)$ in Sec. VI, Eq. 1.

I. INTEGRALS OF THE TYPE $\int x^p e^{-ax} dx$

A.
$$A(p, \alpha) = \int_0^\infty x^p e^{-ax} dx = \frac{\Gamma(p+1)}{\alpha^{p+1}} \quad (1)$$

$$\alpha A(p, \alpha) = p A(p-1, \alpha) \quad (1.1)$$

$$\int_0^\infty e^{-ax} dx = \frac{1}{\alpha} \quad (1.2)$$

$$\int_0^\infty x e^{-ax} dx = \frac{1}{\alpha^2} \quad (1.3)$$

$$\int_0^\infty x^2 e^{-ax} dx = \frac{2}{\alpha^3} \quad (1.4)$$

$$\int_0^\infty x^3 e^{-ax} dx = \frac{6}{\alpha^4} \quad (1.5)$$

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{\alpha^{n+1}} \quad (1.6)$$

B.
$$A_1(p, \alpha, \rho) = \int_0^\rho x^p e^{-ax} dx = \rho^{p+1} \sum_{k=0}^\infty \frac{(-\alpha\rho)^k}{k! (p+k+1)!} = \frac{1}{\alpha^{p+1}} \gamma(p+1, \alpha\rho) \quad (2)$$

$$\alpha A_1(p, \alpha, \rho) = p A_1(p-1, \alpha, \rho) - \rho^p e^{-\alpha\rho} \quad (2.1)$$

$$\int_0^\rho e^{-ax} dx = \rho \sum_{k=0}^\infty \frac{(-\alpha\rho)^k}{k! (k+1)!} = \frac{1}{\alpha} \gamma(1, \alpha\rho) = \frac{1}{\alpha} (1 - e^{-\alpha\rho}) \quad (2.2)$$

$$\int_0^\rho x e^{-ax} dx = \frac{1}{\alpha^2} [1 - e^{-\alpha\rho} (1 + \alpha\rho)] \quad (2.3)$$

$$\int_0^\rho x^2 e^{-ax} dx = \frac{2}{\alpha^3} \left\{ 1 - e^{-\alpha\rho} \left[1 + \alpha\rho + \frac{(\alpha\rho)^2}{2!} \right] \right\} \quad (2.4)$$

$$\int_0^\rho x^3 e^{-ax} dx = \frac{6}{\alpha^4} \left\{ 1 - e^{-\alpha\rho} \left[1 + \alpha\rho + \frac{(\alpha\rho)^2}{2!} + \frac{(\alpha\rho)^3}{3!} \right] \right\} \quad (2.5)$$

$$\int_0^\rho x^n e^{-ax} dx = \frac{n!}{\alpha^{n+1}} [1 - e^{-\alpha\rho} e_n(\alpha\rho)] \quad (2.6)$$

$$A_1(p, 0, \rho) = \int_0^\rho x^p dx = \frac{\rho^{p+1}}{(p+1)} \quad (2.7)$$

C.
$$A_2(p, \alpha, \rho) = \int_\rho^\infty x^p e^{-ax} dx = A(p, \alpha) - A_1(p, \alpha, \rho) = \frac{1}{\alpha^{p+1}} \Gamma(p+1, \alpha\rho) \quad (3)$$

$$\alpha A_2(p, \alpha, \rho) = p A_2(p-1, \alpha, \rho) + \rho^p e^{-\alpha \rho} \quad (3.1)$$

$$\int_{\rho}^{\infty} e^{-\alpha x} dx = \frac{1}{\alpha} e^{-\alpha \rho} \quad (3.2)$$

$$\int_{\rho}^{\infty} x e^{-\alpha x} dx = \frac{1}{\alpha^2} e^{-\alpha \rho} (1 + \alpha \rho) \quad (3.3)$$

$$\int_{\rho}^{\infty} x^2 e^{-\alpha x} dx = \frac{2}{\alpha^3} e^{-\alpha \rho} \left[1 + \alpha \rho + \frac{(\alpha \rho)^2}{2!} \right] \quad (3.4)$$

$$\int_{\rho}^{\infty} x^3 e^{-\alpha x} dx = \frac{6}{\alpha^4} e^{-\alpha \rho} \left[1 + \alpha \rho + \frac{(\alpha \rho)^2}{2!} + \frac{(\alpha \rho)^3}{3!} \right] \quad (3.5)$$

$$\int_{\rho}^{\infty} x^n e^{-\alpha x} dx = \frac{n!}{\alpha^{n+1}} e^{-\alpha \rho} e_n(\alpha \rho) \quad (3.6)$$

II. INTEGRALS OF THE TYPE $\int x^p e^{-\alpha x} \ln x dx$

A.

$$B(p, \alpha) = \int_0^{\infty} x^p e^{-\alpha x} \ln x dx = \frac{\Gamma(p+1)}{\alpha^{p+1}} [\psi(p+1) - \ln \alpha] \quad (1)$$

$$\alpha B(p, \alpha) = p B(p-1, \alpha) + A(p-1, \alpha) \quad (1.1)$$

$$\int_0^{\infty} e^{-\alpha x} \ln x dx = -\frac{1}{\alpha} (\gamma + \ln \alpha) \quad (1.2)$$

$$\int_0^{\infty} x e^{-\alpha x} \ln x dx = -\frac{1}{\alpha^2} (\gamma + \ln \alpha - 1) \quad (1.3)$$

$$\int_0^{\infty} x^2 e^{-\alpha x} \ln x dx = -\frac{2}{\alpha^3} \left(\gamma + \ln \alpha - 1 - \frac{1}{2} \right) \quad (1.4)$$

$$\int_0^{\infty} x^3 e^{-\alpha x} \ln x dx = -\frac{6}{\alpha^4} \left(\gamma + \ln \alpha - 1 - \frac{1}{2} - \frac{1}{3} \right) \quad (1.5)$$

$$\int_0^{\infty} x^n e^{-\alpha x} \ln x dx = -\frac{n!}{\alpha^{n+1}} \left(\gamma + \ln \alpha - 1 - \frac{1}{2} - \cdots - \frac{1}{n} \right) = -\frac{n!}{\alpha^{n+1}} \left(\gamma + \ln \alpha - \sum_{i=1}^n \frac{1}{i} \right) \quad (1.6)$$

$$\int_0^{\infty} e^{-x} \ln x dx = -\gamma \quad (1.7)$$

$$\int_0^{\infty} x^1 e^{-ax} \ln x \, dx = -\frac{\sqrt{\pi}}{\sqrt{4a^3}} (\gamma + \ln a + 2\ln 2 - 2) \quad (1.8)$$

$$B. \quad B_1(p, \alpha, \rho) = \int_0^{\rho} x^p e^{-ax} \ln x \, dx = -\rho^{p+1} \left[\sum_{k=0}^{\infty} \frac{(-\alpha\rho)^k}{k! (p+k+1)^2} - \ln \rho \sum_{k=0}^{\infty} \frac{(-\alpha\rho)^k}{k! (p+k+1)^1} \right] \quad (2)$$

$$\alpha B_1(p, \alpha, \rho) = p B_1(p-1, \alpha, \rho) + A_1(p-1, \alpha, \rho) - \rho^p \ln \rho e^{-\alpha\rho} \quad (2.1)$$

$$\int_0^{\rho} e^{-ax} \ln x \, dx = -\rho \left[\sum_{k=0}^{\infty} \frac{(-\alpha\rho)^k}{k! (k+1)^2} - \ln \rho \sum_{k=0}^{\infty} \frac{(-\alpha\rho)^k}{k! (k+1)^1} \right] = -\frac{1}{\alpha} \left\{ \gamma + \ln \alpha + \ln \rho e^{-\alpha\rho} + [-E_i(-\alpha\rho)] \right\} \quad (2.2)$$

$$\int_0^{\rho} x e^{-ax} \ln x \, dx = -\frac{1}{\alpha^2} \left\{ \gamma + \ln \alpha + (1 + \alpha\rho) \ln \rho e^{-\alpha\rho} + [-E_i(-\alpha\rho)] - 1 + e^{-\alpha\rho} \right\} \quad (2.3)$$

$$\int_0^{\rho} x^2 e^{-ax} \ln x \, dx = -\frac{2}{\alpha^3} \left\{ \gamma + \ln \alpha + \left[1 + \alpha\rho + \frac{(\alpha\rho)^2}{2!} \right] \ln \rho e^{-\alpha\rho} + [-E_i(-\alpha\rho)] - 1 - \frac{1}{2} + e^{-\alpha\rho} \left[1 + \frac{1}{2} + \frac{1}{2} (\alpha\rho) \right] \right\} \quad (2.4)$$

$$\int_0^{\rho} x^3 e^{-ax} \ln x \, dx = -\frac{6}{\alpha^4} \left\{ \gamma + \ln \alpha + \left[1 + \alpha\rho + \frac{(\alpha\rho)^2}{2!} + \frac{(\alpha\rho)^3}{3!} \right] \ln \rho e^{-\alpha\rho} + [-E_i(-\alpha\rho)] - 1 - \frac{1}{2} - \frac{1}{3} + e^{-\alpha\rho} \left[1 + \frac{1}{2} + \frac{1}{3} + (\alpha\rho) \left(\frac{1}{2} + \frac{1}{3} \right) + \frac{(\alpha\rho)^2}{2!} \left(\frac{1}{3} \right) \right] \right\} \quad (2.5)$$

$$\int_0^{\rho} x^n e^{-ax} \ln x \, dx = -\frac{n!}{\alpha^{n+1}} \left\{ \gamma + \ln \alpha + \ln \rho e^{-\alpha\rho} e_n(\alpha\rho) + [-E_i(-\alpha\rho)] + \sum_{i=0}^{n-1} \frac{1}{i+1} [e_i(\alpha\rho) e^{-\alpha\rho} - 1] \right\} \quad (2.6)$$

$$B_1(p, 0, \rho) = \int_0^{\rho} x^p \ln x \, dx = \frac{\rho^{p+1}}{(p+1)} \left(\ln \rho - \frac{1}{p+1} \right) \quad (2.7)$$

$$\sum_{k=0}^{\infty} \frac{(-x)^k}{k! (k+p+1)^1} = \frac{\gamma(p+1, x)}{x^{p+1}} \quad (2.8)$$

$$\sum_{k=0}^{\infty} \frac{(-x)^k}{k! (k+n+1)^1} = \frac{\gamma(n+1, x)}{x^{n+1}} = \frac{n!}{x^{n+1}} [1 - e^{-x} e_n(x)] \quad (2.9)$$

$$\sum_{k=0}^{\infty} \frac{(-x)^k}{k! (n+k+1)^2} = \frac{n!}{x^{n+1}} \left\{ \ln x + [-E_i(-x)] - \psi(n+1) + e^{-x} \sum_{k=0}^{n-1} \frac{e_k(x)}{k+1} \right\} \quad (2.10)$$

$$C. \quad B_2(p, \alpha, \rho) = \int_{\rho}^{\infty} x^p e^{-ax} \ln x \, dx = B(p, \alpha) - B_1(p, \alpha, \rho) \quad (3)$$

$$\alpha B_2(p, \alpha, \rho) = p B_2(p-1, \alpha, \rho) + A_1(p-1, \alpha, \rho) + \rho^p \ln \rho e^{-\alpha\rho} \quad (3.1)$$

$$\int_{\rho}^{\infty} e^{-ax} \ln x \, dx = \frac{1}{\alpha} \left\{ \ln \rho e^{-\alpha\rho} + [-E_i(-\alpha\rho)] \right\} \quad (3.2)$$

$$\int_{\rho}^{\infty} x e^{-ax} \ln x \, dx = \frac{1}{\alpha^2} \left\{ (1 + \alpha\rho) \ln \rho e^{-\alpha\rho} + [-E_i(-\alpha\rho)] + e^{-\alpha\rho} \right\} \quad (3.3)$$

$$\int_{\rho}^{\infty} x^2 e^{-ax} \ln x \, dx = \frac{2}{\alpha^3} \left\{ \left[1 + \alpha\rho + \frac{(\alpha\rho)^2}{2!} \right] \ln \rho e^{-a\rho} + [-E_i(-\alpha\rho)] + e^{-a\rho} \left[1 + \frac{1}{2} + \frac{1}{2} (\alpha\rho) \right] \right\} \quad (3.4)$$

$$\begin{aligned} \int_{\rho}^{\infty} x^3 e^{-ax} \ln x \, dx = \frac{6}{\alpha^4} \left\{ \left[1 + \alpha\rho + \frac{(\alpha\rho)^2}{2!} + \frac{(\alpha\rho)^3}{3!} \right] \ln \rho e^{-a\rho} + [-E_i(-\alpha\rho)] \right. \\ \left. + e^{-a\rho} \left[1 + \frac{1}{2} + \frac{1}{3} + (\alpha\rho) \left(\frac{1}{2} + \frac{1}{3} \right) + \frac{(\alpha\rho)^2}{2!} \left(\frac{1}{3} \right) \right] \right\} \end{aligned} \quad (3.5)$$

$$\int_{\rho}^{\infty} x^n e^{-ax} \ln x \, dx = \frac{n!}{\alpha^{n+1}} \left\{ \ln \rho e^{-a\rho} e_n(\alpha\rho) + [-E_i(-\alpha\rho)] + \sum_{i=0}^{n-1} \frac{e_i(\alpha\rho)}{(i+1)} e^{-a\rho} \right\} \quad (3.6)$$

III. INTEGRALS OF THE TYPE $\int x^p e^{-ax} \ln^2 x \, dx$

$$A. \quad C(p, \alpha) = \int_0^{\infty} x^p e^{-ax} \ln^2 x \, dx = \frac{\Gamma(p+1)}{\alpha^{p+1}} \left\{ [\psi(p+1) - \ln \alpha]^2 + \zeta(2, p+1) \right\} \quad (1)$$

$$\alpha C(p, \alpha) = p C(p-1, \alpha) + 2B(p-1, \alpha) \quad (1.1)$$

$$\int_0^{\infty} e^{-ax} \ln^2 x \, dx = \frac{1}{\alpha} [(\gamma + \ln \alpha)^2 + \zeta(2)] \quad (1.2)$$

$$\int_0^{\infty} x e^{-ax} \ln^2 x \, dx = \frac{1}{\alpha^2} [(\gamma + \ln \alpha - 1)^2 + \zeta(2) - 1] \quad (1.3)$$

$$\int_0^{\infty} x^2 e^{-ax} \ln^2 x \, dx = \frac{2}{\alpha^3} \left[\left(\gamma + \ln \alpha - 1 - \frac{1}{2} \right)^2 + \zeta(2) - 1 - \frac{1}{2^2} \right] \quad (1.4)$$

$$\int_0^{\infty} x^3 e^{-ax} \ln^2 x \, dx = \frac{6}{\alpha^4} \left[\left(\gamma + \ln \alpha - 1 - \frac{1}{2} - \frac{1}{3} \right)^2 + \zeta(2) - 1 - \frac{1}{2^2} - \frac{1}{3^2} \right] \quad (1.5)$$

$$\int_0^{\infty} x^n e^{-ax} \ln^2 x \, dx = \frac{n!}{\alpha^{n+1}} \left[\left(\gamma + \ln \alpha - \sum_{i=0}^{n-1} \frac{1}{i+1} \right)^2 + \zeta(2) - \sum_{i=0}^{n-1} \frac{1}{(i+1)^2} \right] \quad (1.6)$$

$$\zeta(2, n+1) = \zeta(2) - \sum_{i=0}^{n-1} \frac{1}{(i+1)^2} \quad (1.7)$$

$$\zeta(2) = \frac{\pi^2}{6} = 1.644\,934\,0668 \dots \quad (1.8)$$

$$\psi(n+1) = -\gamma + 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \quad (1.9)$$

B.

$$C_1(p, \alpha, \rho) = \int_0^\rho x^p e^{-\alpha x} \ln^2 x \, dx = 2\rho^{p+1} \left[\sum_{k=0}^{\infty} \frac{(-\alpha\rho)^k}{k! (p+k+1)^3} - \ln\rho \sum_{k=0}^{\infty} \frac{(-\alpha\rho)^k}{k! (p+k+1)^2} + \frac{1}{2} \ln^2 \rho \sum_{k=0}^{\infty} \frac{(-\alpha\rho)^k}{k! (p+k+1)} \right] \quad (2)$$

$$\alpha C_1(p, \alpha, \rho) = p C_1(p-1, \alpha, \rho) + 2B_1(p-1, \alpha, \rho) - \rho^p \ln^2 \rho e^{-\alpha\rho} \quad (2.1)$$

$$\int_0^\rho e^{-\alpha x} \ln^2 x \, dx = 2\rho \sum_{k=0}^{\infty} \frac{(-\alpha\rho)^k}{k! (k+1)^3} - \frac{2 \ln\rho}{\alpha} \left\{ \gamma + \ln\alpha + \frac{1}{2} \ln\rho (1 + e^{-\alpha\rho}) + [-E_i(-\alpha\rho)] \right\} \quad (2.2)$$

$$\int_0^\rho x e^{-\alpha x} \ln^2 x \, dx = 2\rho^2 \sum_{k=0}^{\infty} \frac{(-\alpha\rho)^k}{k! (k+2)^3} - \frac{2 \ln\rho}{\alpha^2} \left\{ \gamma + \ln\alpha + \frac{1}{2} \ln\rho [1 + (1 + \alpha\rho)e^{-\alpha\rho}] + [-E_i(-\alpha\rho)] + e^{-\alpha\rho} - 1 \right\} \quad (2.3)$$

$$\begin{aligned} \int_0^\rho x^2 e^{-\alpha x} \ln^2 x \, dx &= 2\rho^3 \sum_{k=0}^{\infty} \frac{(-\alpha\rho)^k}{k! (k+3)^3} - \frac{4 \ln\rho}{\alpha^3} \left\{ \gamma + \ln\alpha + \frac{1}{2} \ln\rho \left\{ 1 + \left[1 + \alpha\rho + \frac{(\alpha\rho)^2}{2!} \right] e^{-\alpha\rho} \right\} + [-E_i(-\alpha\rho)] \right. \\ &\quad \left. + e^{-\alpha\rho} \left[1 + \frac{1}{2} + (\alpha\rho) \frac{1}{2} \right] - 1 - \frac{1}{2} \right\} \end{aligned} \quad (2.4)$$

$$\begin{aligned} \int_0^\rho x^3 e^{-\alpha x} \ln^2 x \, dx &= 2\rho^4 \sum_{k=0}^{\infty} \frac{(-\alpha\rho)^k}{k! (k+4)^3} - \frac{12 \ln\rho}{\alpha^4} \left\{ \gamma + \ln\alpha + \frac{1}{2} \ln\rho \left\{ 1 + \left[1 + \alpha\rho + \frac{(\alpha\rho)^2}{2!} + \frac{(\alpha\rho)^3}{3!} \right] e^{-\alpha\rho} \right\} \right. \\ &\quad \left. + [-E_i(-\alpha\rho)] + e^{-\alpha\rho} \left[1 + \frac{1}{2} + \frac{1}{3} + (\alpha\rho) \left(\frac{1}{2} + \frac{1}{3} \right) + \frac{(\alpha\rho)^2}{2!} \left(\frac{1}{3} \right) \right] - 1 - \frac{1}{2} - \frac{1}{3} \right\} \end{aligned} \quad (2.5)$$

$$\begin{aligned} \int_0^\rho x^n e^{-\alpha x} \ln^2 x \, dx &= 2\rho^{n+1} \sum_{k=0}^{\infty} \frac{(-\alpha\rho)^k}{k! (n+k+1)^3} - \frac{2n! \ln\rho}{\alpha^{n+1}} \left\{ \gamma + \ln\alpha + \frac{1}{2} \ln\rho [1 + e_n(\alpha\rho)e^{-\alpha\rho}] + [-E_i(-\alpha\rho)] \right. \\ &\quad \left. + \sum_{i=0}^{n-1} \frac{1}{(i+1)} [e_i(\alpha\rho)e^{-\alpha\rho} - 1] \right\} \end{aligned} \quad (2.6)$$

$$C_1(p, 0, \rho) = \int_0^\rho x^p \ln^2 x \, dx = \frac{\rho^{p+1}}{p+1} \left[\ln^2 \rho - \frac{2}{(p+1)} \ln\rho + \frac{2}{(p+1)^2} \right] \quad (2.7)$$

$$S(p, x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k! (p+k+1)^3} \quad (2.8)$$

$$xS(n+1, x) = (n+1) S(n, x) - \frac{n!}{x^{n+1}} \left\{ \gamma + \ln x + [-E_i(-x)] + \sum_{i=0}^{n-1} \frac{1}{(i+1)} [e_i(x)e^{-x} - 1] \right\} \quad (2.9)$$

C.

$$C_2(p, \alpha, \rho) = \int_p^\infty x^p e^{-\alpha x} \ln^2 x \, dx = C(p, \alpha) - C_1(p, \alpha, \rho) \quad (3)$$

$$\alpha C_2(p, \alpha, \rho) = p C_2(p-1, \alpha, \rho) + 2B_2(p-1, \alpha, \rho) + \rho^p e^{-\alpha\rho} \ln^2 \rho \quad (3.1)$$

$$\int_p^\infty e^{-\alpha x} \ln^2 x \, dx = \frac{1}{\alpha} \left\{ (\gamma + \ln\alpha\rho)^2 + \zeta(2) + \ln^2 \rho e^{-\alpha\rho} + 2 \ln\rho [-E_i(-\alpha\rho)] \right\} - 2\rho \sum_{k=0}^{\infty} \frac{(-\alpha\rho)^k}{k! (k+1)^3} \quad (3.2)$$

$$\int_{\rho}^{\infty} x e^{-ax} \ln^2 x \, dx = \frac{1}{\alpha^2} \left\{ (\gamma + \ln \alpha \rho - 1)^2 + \zeta(2) - 1 + \ln^2 \rho (1 + \alpha \rho) e^{-\alpha \rho} + 2 \ln \rho e^{-\alpha \rho} + 2 \ln \rho [-E_i(-\alpha \rho)] \right\} \\ - 2 \rho^2 \sum_{k=0}^{\infty} \frac{(-\alpha \rho)^k}{k! (k+2)^3} \quad (3.3)$$

$$\int_{\rho}^{\infty} x^2 e^{-ax} \ln^2 x \, dx = \frac{2}{\alpha^3} \left\{ \left(\gamma + \ln \alpha \rho - 1 - \frac{1}{2} \right)^2 + \zeta(2) - 1 - \frac{1}{2^2} + \ln^2 \rho e_2(\alpha \rho) e^{-\alpha \rho} + 2 \ln \rho e^{-\alpha \rho} \left[1 + \frac{1}{2} + (\alpha \rho) \frac{1}{2} \right] \right. \\ \left. + 2 \ln \rho [-E_i(-\alpha \rho)] \right\} - 2 \rho^3 \sum_{k=0}^{\infty} \frac{(-\alpha \rho)^k}{k! (k+3)^3} \quad (3.4)$$

$$\int_{\rho}^{\infty} x^3 e^{-ax} \ln^2 x \, dx = \frac{6}{\alpha^4} \left\{ \left(\gamma + \ln \alpha \rho - 1 - \frac{1}{2} - \frac{1}{3} \right)^2 + \zeta(2) - 1 - \frac{1}{2^2} - \frac{1}{3^2} + \ln^2 \rho e_3(\alpha \rho) e^{-\alpha \rho} + 2 \ln \rho e^{-\alpha \rho} \right. \\ \left. \times \left[1 + \frac{1}{2} + \frac{1}{3} + (\alpha \rho) \left(\frac{1}{2} + \frac{1}{3} \right) + \frac{(\alpha \rho)^2}{2!} \left(\frac{1}{3} \right) \right] + 2 \ln \rho [-E_i(-\alpha \rho)] \right\} - 2 \rho^4 \sum_{k=0}^{\infty} \frac{(-\alpha \rho)^k}{k! (k+4)^3} \quad (3.5)$$

$$\int_{\rho}^{\infty} x^n e^{-ax} \ln^2 x \, dx = \frac{n!}{\alpha^{n+1}} \left\{ \left(\gamma + \ln \alpha \rho - \sum_{i=0}^{n-1} \frac{1}{i+1} \right)^2 + \zeta(2) - \sum_{i=0}^{n-1} \frac{1}{(i+1)^2} + \ln^2 \rho e_n(\alpha \rho) e^{-\alpha \rho} + 2 \ln \rho e^{-\alpha \rho} \right. \\ \left. \times \sum_{i=0}^{n-1} \frac{e_i(\alpha \rho)}{(i+1)} + 2 \ln \rho [-E_i(-\alpha \rho)] \right\} - 2 \rho^{n+1} \sum_{k=0}^{\infty} \frac{(-\alpha \rho)^k}{k! (k+n+1)^3} \quad (3.6)$$

$$\int_1^{\infty} x^n e^{-ax} \ln^2 x \, dx = \frac{n!}{\alpha^{n+1}} \left\{ [\psi(n+1) - \ln \alpha]^2 + \zeta(2, n+1) \right\} - 2 \sum_{k=0}^{\infty} \frac{(-\alpha)^k}{k! (k+n+1)^3} \quad (3.7)$$

IV. INTEGRALS OF THE TYPE $\int x^p e^{-ax} \ln^3 x \, dx$

A.

$$D(p, \alpha) = \int_0^{\infty} x^p e^{-ax} \ln^3 x \, dx = \frac{\Gamma(p+1)}{\alpha^{p+1}} \left\{ [\psi(p+1) - \ln \alpha]^3 + 3\zeta(2, p+1) [\psi(p+1) - \ln \alpha] - 2\zeta(3, p+1) \right\} \quad (1)$$

$$\alpha D(p, \alpha) = p D(p-1, \alpha) + 6C(p-1, \alpha) \quad (1.1)$$

$$\int_0^{\infty} e^{-ax} \ln^3 x \, dx = -\frac{1}{\alpha} [(\gamma + \ln \alpha)^3 + 3\zeta(2)(\gamma + \ln \alpha) + 2\zeta(3)] \quad (1.2)$$

$$\int_0^{\infty} x e^{-ax} \ln^3 x \, dx = -\frac{1}{\alpha^2} [(\gamma + \ln \alpha - 1)^3 + 3\zeta(2, 2)(\gamma + \ln \alpha - 1) + 2\zeta(3, 2)] \quad (1.3)$$

$$\int_0^{\infty} x^2 e^{-ax} \ln^3 x \, dx = -\frac{2}{\alpha^3} \left[\left(\gamma + \ln \alpha - 1 - \frac{1}{2} \right)^3 + 3\zeta(2, 3) \left(\gamma + \ln \alpha - 1 - \frac{1}{2} \right) + 2\zeta(3, 3) \right] \quad (1.4)$$

$$\int_0^\infty x^3 e^{-ax} \ln^3 x \, dx = -\frac{6}{a^4} \left[\left(\gamma + \ln a - 1 - \frac{1}{2} - \frac{1}{3} \right)^3 + 3\zeta(2,4) \left(\gamma + \ln a - 1 - \frac{1}{2} - \frac{1}{3} \right) + 2\zeta(3,4) \right] \quad (1.5)$$

$$\int_0^\infty x^n e^{-ax} \ln^3 x \, dx = -\frac{n!}{a^{n+1}} \left[\left(\gamma + \ln a - \sum_{i=0}^{n-1} \frac{1}{i+1} \right)^3 + 3\zeta(2,n+1) \left(\gamma + \ln a - \sum_{i=0}^{n-1} \frac{1}{i+1} \right) + 2\zeta(3,n+1) \right] \quad (1.6)$$

$$\zeta(3,n+1) = \zeta(3) - \sum_{i=0}^{n-1} \frac{1}{(i+1)^3} \quad (1.7)$$

$$\zeta(3) = \frac{\pi^3}{25.794 \, 36 \dots} = 1.202 \, 056 \, 9032 \dots \quad (1.8)$$

B.
$$D_1(p, \alpha, \rho) = \int_0^\rho x^p e^{-ax} \ln^3 x \, dx = -6\rho^{p+1} \left[\sum_{k=0}^\infty \frac{(-\alpha\rho)^k}{k!(k+p+1)^4} - \ln\rho \sum_{k=0}^\infty \frac{(-\alpha\rho)^k}{k!(k+p+1)^3} \right. \\ \left. + \frac{\ln^2 \rho}{2} \sum_{k=0}^\infty \frac{(-\alpha\rho)^k}{k!(k+p+1)^2} - \frac{\ln^3 \rho}{6} \sum_{k=0}^\infty \frac{(-\alpha\rho)^k}{k!(k+p+1)} \right] \quad (2)$$

$$\alpha D_1(p, \alpha, \rho) = p D_1(p-1, \alpha, \rho) + 6C_1(p-1, \alpha, \rho) - \rho^p e^{-\alpha\rho} \ln^3 \rho \quad (2.1)$$

$$\int_0^\rho e^{-ax} \ln^3 x \, dx = -6\rho \left[\sum_{k=0}^\infty \frac{(-\alpha\rho)^k}{k!(k+1)^4} - \ln\rho \sum_{k=0}^\infty \frac{(-\alpha\rho)^k}{k!(k+1)^3} \right] - \frac{3 \ln^2 \rho}{\alpha} \left\{ \gamma + \ln \alpha\rho + [-E_i(-\alpha\rho)] + \frac{1}{3} \ln\rho (1 - e^{-\alpha\rho}) \right\} \quad (2.2)$$

$$D_1(p, 0, \rho) = \int_0^\rho x^p \ln^3 x \, dx = \frac{\rho^{p+1}}{(p+1)} \left[\ln^3 \rho - \frac{3}{(p+1)} \ln^2 \rho + \frac{6}{(p+1)^2} \ln \rho - \frac{6}{(p+1)^3} \right] \quad (2.3)$$

C.
$$D_2(p, \alpha, \rho) = \int_\rho^\infty x^p e^{-ax} \ln^3 x \, dx = D(p, \alpha) - D_1(p, \alpha, \rho) \quad (3)$$

$$\alpha D_2(p, \alpha, \rho) = p D_2(p-1, \alpha, \rho) + 6C_2(p-1, \alpha, \rho) + \rho^p e^{-\alpha\rho} \ln^3 \rho \quad (3.1)$$

$$\int_\rho^\infty e^{-ax} \ln^3 x \, dx = 6\rho \left[\sum_{k=0}^\infty \frac{(-\alpha\rho)^k}{k!(k+1)^4} - \ln\rho \sum_{k=0}^\infty \frac{(-\alpha\rho)^k}{k!(k+1)^3} \right] + \frac{3 \ln^2 \rho}{\alpha} \left\{ \ln\rho + [-E_i(-\alpha\rho)] + \frac{1}{3} \ln\rho (1 - e^{-\alpha\rho}) \right\} \\ - \frac{1}{\alpha} \{ (\gamma + \ln \alpha)^3 + 3(\gamma + \ln \alpha) [\zeta(2) - \ln^2 \rho] + 2\zeta(3) \} \quad (3.2)$$

V. MISCELLANEOUS RELATED INTEGRALS

A.
$$F(n, \alpha, p) = \int_0^\infty x^p e^{-\alpha x} (\ln x)^n dx \quad (1)$$

$$\alpha F(n, \alpha, p) = p F(n, \alpha, p-1) + n F(n-1, \alpha, p-1) \quad (1.1)$$

$$\begin{aligned} F(n, \alpha, p) = & \sum_{k=0}^{n-2} \left[\sum_{l=k}^{n-2} \binom{n-1}{l} \binom{l}{k} (\ln \alpha)^{l-k} (-1)^{n-l} (n-1-l)! \zeta(n-l, p+1) \right] F(k, \alpha, p) \\ & + \sum_{k=0}^{n-1} \binom{n-1}{k} (\ln \alpha)^{n-k-1} \left[\psi(p+1) - \frac{n}{n-k} \ln \alpha \right] F(k, \alpha, p) \end{aligned} \quad (1.2)$$

$$\begin{aligned} F(n, \alpha, p) = & \frac{\Gamma(p+1)}{\alpha^{p+1}} \left[\theta^n + 2! \binom{n}{2} \left(\frac{\Phi_2}{2} \right) \theta^{n-2} - 3! \binom{n}{3} \left(\frac{\Phi_3}{3} \right) \theta^{n-3} + 4! \binom{n}{4} \left(\frac{\Phi_4}{4} + \frac{\Phi_2^2}{8} \right) \theta^{n-4} \right. \\ & - 5! \binom{n}{5} \left(\frac{\Phi_5}{5} + \frac{\Phi_3 \Phi_2}{6} \right) \theta^{n-5} + 6! \binom{n}{6} \left(\frac{\Phi_6}{6} + \frac{\Phi_4 \Phi_2}{8} + \frac{\Phi_3^2}{18} + \frac{\Phi_2^3}{48} \right) \theta^{n-6} - 7! \binom{n}{7} \\ & \left. \times \left(\frac{\Phi_7}{7} + \frac{\Phi_5 \Phi_2}{10} + \frac{\Phi_4 \Phi_3}{12} + \frac{\Phi_3 \Phi_2^2}{24} \right) \theta^{n-7} + \dots \right] \end{aligned} \quad (1.3)$$

where $\theta = \psi(p+1) - \ln \alpha$

$$\Phi_r = \zeta(r, p+1)$$

$$\begin{aligned} F(n, \alpha, 0) = & \sum_{k=0}^{n-2} \left[\sum_{l=k}^{n-2} \binom{n-1}{l} \binom{l}{k} (\ln \alpha)^{l-k} (-1)^{n-l} (n-l-1)! \zeta(n-l) \right] F(k, \alpha, 0) \\ & - \sum_{k=0}^{n-1} \binom{n-1}{k} (\ln \alpha)^{n-k-1} \left(\gamma + \frac{n}{n-k} \ln \alpha \right) F(k, \alpha, 0) \end{aligned} \quad (1.4)$$

$$F(n, 1, 0) = (-1)^n (n-1)! \sum_{k=0}^{n-2} \frac{(-1)^k \zeta(n-k)}{k!} F(k, 1, 0) - \gamma F(n-1, 1, 0) \quad (1.5)$$

B.
$$F_1(n, \alpha, p, \rho) = \int_0^\rho x^p e^{-\alpha x} (\ln x)^n dx = (-1)^n n! \rho^{p+1} \sum_{k=0}^n \frac{(-\ln \rho)^k}{k!} \left[\sum_{l=0}^\infty \frac{(-\alpha \rho)^l}{l! (p+l+1)^{n-k+1}} \right] \quad (2)$$

$$\alpha F_1(n, \alpha, p, \rho) = p F_1(n, \alpha, p-1, \rho) + n F_1(n-1, \alpha, p-1, \rho) - \rho^p e^{-\alpha \rho} (\ln \rho)^n \quad (2.1)$$

$$F_1(n, 0, p, \rho) = \int_0^\rho x^p (\ln x)^n dx = \frac{\rho^{p+1}}{(p+1)} \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} (\ln \rho)^{n-k}}{(p+1)^k} \quad (2.2)$$

C.
$$\int_\rho^\infty e^{-\alpha x} (\ln x)^n dx = \frac{(\ln \rho)^n e^{-\alpha \rho}}{\alpha} + \frac{n}{\alpha} \int_\rho^\infty \frac{e^{-\alpha x} (\ln x)^{n-1}}{x} dx \quad (3)$$

$$\int_\rho^\infty e^{-\alpha x} \ln x dx = \frac{\ln \rho e^{-\alpha \rho}}{\alpha} + \frac{1}{\alpha} \int_\rho^\infty \frac{e^{-\alpha x}}{x} dx = \frac{1}{\alpha} \left\{ \ln \rho e^{-\alpha \rho} + [-E_i(-\alpha \rho)] \right\} \quad (3.1)$$

(See Sec. II, Eq. 3.2.)

$$\int_\rho^\infty e^{-\alpha x} \ln^2 x dx = \frac{\ln^2 \rho e^{-\alpha \rho}}{\alpha} + \frac{2}{\alpha} \int_\rho^\infty \frac{e^{-\alpha x} \ln x}{x} dx \quad (3.2)$$

$$\int_p^\infty \frac{\ln x e^{-ax}}{x} dx = \frac{1}{2} \left\{ (\gamma + \ln a + \ln p)^2 + \zeta(2) + 2 \ln p [-E_i(-\alpha p)] \right\} + \sum_{k=0}^{\infty} \frac{(-\alpha p)^{k+1}}{k! (k+1)^3} \quad (3.3)$$

(See Sec. III, Eq. 3.2 and Sec. V, Eq. 3.2.)

$$\int_p^\infty \frac{\ln x e^{-ax}}{x} dx = [-E_i(-\alpha p)] \ln p + \int_p^\infty \frac{[-E_i(-\alpha x)]}{x} dx \quad (3.4)$$

$$\int_p^\infty \frac{[-E_i(-\alpha x)]}{x} dx = \frac{1}{2} [(\gamma + \ln a + \ln p)^2 + \zeta(2)] + \sum_{k=0}^{\infty} \frac{(-\alpha p)^{k+1}}{k! (k+1)^3} \quad (3.5)$$

(See Sec. V, Eq. 3.3 and 3.4.)

VI. INTEGRALS OF THE TYPE $\int x^p [-E_i(-\beta x)] e^{-ax} dx$

$$\begin{aligned} \text{A. } G(p, \alpha, \beta) &= \int_0^\infty x^p e^{-ax} [-E_i(-\beta x)] dx = \frac{\Gamma(p+1)}{(p+1)(\alpha+\beta)^{p+1}} {}_2F_1\left(1, p+1; p+2; \frac{\alpha}{\alpha+\beta}\right) \\ &= \frac{\Gamma(p+1)}{\alpha^{p+1}} B_{\frac{\alpha}{\alpha+\beta}}(p+1, 0) \\ &= \frac{\Gamma(p+1)}{(\alpha+\beta)^{p+1}} \Phi\left(\frac{\alpha}{\alpha+\beta}, 1, p+1\right) \\ &= \frac{\Gamma(p+1)}{(\alpha+\beta)^{p+1}} \sum_{k=0}^{\infty} \frac{1}{(p+k+1)} \left(\frac{\alpha}{\alpha+\beta}\right)^k \end{aligned} \quad (1)$$

$$\alpha G(p, \alpha, \beta) = p G(p-1, \alpha, \beta) - A(p-1, \alpha+\beta) \quad (1.1)$$

$$\int_0^\infty e^{-ax} [-E_i(-\beta x)] dx = \frac{1}{\alpha} \ln\left(1 + \frac{\alpha}{\beta}\right) \quad (1.2)$$

$$\int_0^\infty x e^{-ax} [-E_i(-\beta x)] dx = \frac{1}{\alpha^2} \left[\ln\left(1 + \frac{\alpha}{\beta}\right) - \frac{\alpha}{\alpha+\beta} \right] \quad (1.3)$$

$$\int_0^\infty x^2 e^{-ax} [-E_i(-\beta x)] dx = \frac{2}{\alpha^3} \left[\ln\left(1 + \frac{\alpha}{\beta}\right) - 1\left(\frac{\alpha}{\alpha+\beta}\right) - \frac{1}{2}\left(\frac{\alpha}{\alpha+\beta}\right)^2 \right] \quad (1.4)$$

$$\int_0^\infty x^3 e^{-ax} [-E_i(-\beta x)] dx = \frac{6}{\alpha^4} \left[\ln\left(1 + \frac{\alpha}{\beta}\right) - 1\left(\frac{\alpha}{\alpha+\beta}\right) - \frac{1}{2}\left(\frac{\alpha}{\alpha+\beta}\right)^2 - \frac{1}{3}\left(\frac{\alpha}{\alpha+\beta}\right)^3 \right] \quad (1.5)$$

$$\int_0^\infty x^n e^{-ax} [-E_i(-\beta x)] dx = \frac{n!}{\alpha^{n+1}} \left[\ln\left(1 + \frac{\alpha}{\beta}\right) - \sum_{k=1}^n \frac{1}{k} \left(\frac{\alpha}{\alpha+\beta}\right)^k \right] \quad (1.6)$$

$$\int_0^\infty x^p [-E_i(-\beta x)] dx = \frac{\Gamma(p+1)}{(p+1)\beta^{p+1}} \quad (1.7)$$

$$\Phi(z, 1, n+1) = -\frac{1}{z^{n+1}} \left[\ln(1-z) + \sum_{i=1}^\infty \frac{z^i}{i} \right] \quad (1.8)$$

B.

$$\begin{aligned} G_1(p, \alpha, \beta, \rho) &= \int_0^\rho x^p e^{-\alpha x} [-E_i(-\beta x)] dx = \frac{[-E_i(-\beta \rho)] \gamma(p+1, \alpha \rho)}{\alpha^{p+1}} + \frac{1}{\beta^{p+1}} \sum_{k=0}^\infty \frac{\left(-\frac{\alpha}{\beta}\right)^k \gamma(p+k+1, \beta \rho)}{k! (p+k+1)} \\ &= \frac{[-E_i(-\beta \rho)] \gamma(p+1, \alpha \rho)}{\alpha^{p+1}} + \rho^{p+1} \sum_{k=0}^\infty \frac{(-\alpha \rho)^k}{k! (p+k+1)} \sum_{l=0}^\infty \frac{(-\beta \rho)^l}{l! (p+k+l+1)} \end{aligned} \quad (2)$$

$$\alpha G_1(p, \alpha, \beta, \rho) = p G_1(p-1, \alpha, \beta, \rho) - A_1(p-1, \alpha + \beta, \rho) - \rho^p e^{-\alpha \rho} [-E_i(-\beta \rho)] \quad (2.1)$$

$$\int_0^\rho e^{-\alpha x} [-E_i(-\beta x)] dx = \frac{1}{\alpha} \ln\left(1 + \frac{\alpha}{\beta}\right) - \frac{1}{\alpha} \left(e^{-\alpha \rho} [-E_i(-\beta \rho)] - \left\{ -E_i[-(\alpha + \beta)\rho] \right\} \right) \quad (2.2)$$

$$\begin{aligned} \int_0^\rho x e^{-\alpha x} [-E_i(-\beta x)] dx &= \frac{1}{\alpha^2} \left(\ln\left(1 + \frac{\alpha}{\beta}\right) - e^{-\alpha \rho} (1 + \alpha \rho) [-E_i(-\beta \rho)] \right. \\ &\quad \left. + \left\{ -E_i[-(\alpha + \beta)\rho] \right\} + \left(\frac{\alpha}{\alpha + \beta} \right) [e^{-(\alpha + \beta)\rho} - 1] \right) \end{aligned} \quad (2.3)$$

$$\begin{aligned} \int_0^\rho x^2 e^{-\alpha x} [-E_i(-\beta x)] dx &= \frac{2}{\alpha^3} \left(\ln\left(1 + \frac{\alpha}{\beta}\right) - e^{-\alpha \rho} e_2(\alpha \rho) [-E_i(-\beta \rho)] + \left\{ -E_i[-(\alpha + \beta)\rho] \right\} \right. \\ &\quad \left. + \left(\frac{\alpha}{\alpha + \beta} \right) [e^{-(\alpha + \beta)\rho} - 1] + \frac{1}{2} \left(\frac{\alpha}{\alpha + \beta} \right)^2 \left\{ e^{-(\alpha + \beta)\rho} [1 + (\alpha + \beta)\rho] - 1 \right\} \right) \end{aligned} \quad (2.4)$$

$$\begin{aligned} \int_0^\rho x^3 e^{-\alpha x} [-E_i(-\beta x)] dx &= \frac{6}{\alpha^4} \left(\ln\left(1 + \frac{\alpha}{\beta}\right) - e^{-\alpha \rho} e_3(\alpha \rho) [-E_i(-\beta \rho)] + \left\{ -E_i[-(\alpha + \beta)\rho] \right\} \right. \\ &\quad \left. + \left(\frac{\alpha}{\alpha + \beta} \right) [e^{-(\alpha + \beta)\rho} - 1] + \frac{1}{2} \left(\frac{\alpha}{\alpha + \beta} \right)^2 \left\{ e^{-(\alpha + \beta)\rho} e_1[(\alpha + \beta)\rho] - 1 \right\} \right. \\ &\quad \left. + \frac{1}{3} \left(\frac{\alpha}{\alpha + \beta} \right)^3 \left\{ e^{-(\alpha + \beta)\rho} e_2[(\alpha + \beta)\rho] - 1 \right\} \right) \end{aligned} \quad (2.5)$$

$$\begin{aligned} \int_0^\rho x^n e^{-\alpha x} [-E_i(-\beta x)] dx &= \frac{n!}{\alpha^{n+1}} \left(\ln\left(1 + \frac{\alpha}{\beta}\right) - e^{-\alpha \rho} e_n(\alpha \rho) [-E_i(-\beta \rho)] + \left\{ -E_i[-(\alpha + \beta)\rho] \right\} \right. \\ &\quad \left. + \sum_{k=1}^n \frac{1}{k} \left(\frac{\alpha}{\alpha + \beta} \right)^k \left\{ e^{-(\alpha + \beta)\rho} e_{k-1}[(\alpha + \beta)\rho] - 1 \right\} \right) \end{aligned} \quad (2.6)$$

$$\int_0^\rho x^p [-E_i(-\beta x)] dx = \frac{1}{(p+1)} \left\{ \frac{1}{\beta^{p+1}} \gamma(p+1, \beta \rho) + \rho^{p+1} [-E_i(-\beta \rho)] \right\} \quad (2.7)$$

$$C. \quad G_2(p, \alpha, \beta, \rho) = \int_{\rho}^{\infty} x^p e^{-\alpha x} [-E_i(-\beta x)] dx = G(p, \alpha, \beta) - G_1(p, \alpha, \beta, \rho) \quad (3)$$

$$\alpha G_2(p, \alpha, \beta, \rho) = p G_2(p-1, \alpha, \beta, \rho) - A_2(p-1, \alpha+\beta, \rho) + \rho^p e^{-\alpha \rho} [-E_i(-\beta \rho)] \quad (3.1)$$

$$\int_{\rho}^{\infty} e^{-\alpha x} [-E_i(-\beta x)] dx = \frac{1}{\alpha} \left(e^{-\alpha \rho} [-E_i(-\beta \rho)] - \left\{ -E_i[-(\alpha+\beta)\rho] \right\} \right) \quad (3.2)$$

$$\int_{\rho}^{\infty} x e^{-\alpha x} [-E_i(-\beta x)] dx = \frac{1}{\alpha^2} \left(e_1(\alpha \rho) e^{-\alpha \rho} [-E_i(-\alpha \rho)] - \left\{ -E_i[-(\alpha+\beta)\rho] \right\} \right) - \frac{e^{-(\alpha+\beta)\rho}}{\alpha(\alpha+\beta)} \quad (3.3)$$

$$\begin{aligned} \int_{\rho}^{\infty} x^2 e^{-\alpha x} [-E_i(-\beta x)] dx &= \frac{2}{\alpha^3} \left(e_2(\alpha \rho) e^{-\alpha \rho} [-E_i(-\alpha \rho)] - \left\{ -E_i[-(\alpha+\beta)\rho] \right\} \right) \\ &\quad - \frac{e^{-(\alpha+\beta)\rho}}{\alpha^2(\alpha+\beta)^2} [2(\alpha+\beta) + \alpha + \rho \alpha(\alpha+\beta)] \end{aligned} \quad (3.4)$$

$$\begin{aligned} \int_{\rho}^{\infty} x^3 e^{-\alpha x} [-E_i(-\beta x)] dx &= \frac{6}{\alpha^4} \left(e_3(\alpha \rho) e^{-\alpha \rho} [-E_i(-\beta \rho)] - \left\{ -E_i[-(\alpha+\beta)\rho] \right\} \right) \\ &\quad - e^{-(\alpha+\beta)\rho} \sum_{k=1}^3 \frac{e_{k-1}[(\alpha+\beta)\rho]}{k \left(1 + \frac{\beta}{\alpha}\right)^k} \end{aligned} \quad (3.5)$$

$$\begin{aligned} \int_{\rho}^{\infty} x^n e^{-\alpha x} [-E_i(-\beta x)] dx &= \frac{n!}{\alpha^{n+1}} \left(e_n(\alpha \rho) e^{-\alpha \rho} [-E_i(-\beta \rho)] - \left\{ -E_i[-(\alpha+\beta)\rho] \right\} \right) \\ &\quad - e^{-(\alpha+\beta)\rho} \sum_{k=1}^n \frac{e_{k-1}[(\alpha+\beta)\rho]}{k \left(1 + \frac{\beta}{\alpha}\right)^k} \end{aligned} \quad (3.6)$$

$$\int_{\rho}^{\infty} x^p [-E_i(-\beta x)] dx = \frac{1}{(p+1)} \left\{ \frac{1}{\beta^{p+1}} \Gamma(p+1, \beta \rho) - \rho^{p+1} [-E_i(-\beta \rho)] \right\} \quad (3.7)$$

VII. INTEGRALS OF THE TYPE $\int x^p e^{-\alpha x} \ln x [-E_i(-\beta x)] dx$

$$\begin{aligned} A. \quad H(p, \alpha, \beta) &= \int_0^{\infty} x^p e^{-\alpha x} \ln x [-E_i(-\beta x)] dx = \frac{\Gamma(p+1)}{(\alpha+\beta)^{p+1}} \left\{ [\psi(p+1) - \ln(\alpha+\beta)] \right. \\ &\quad \times \left. \Phi\left(\frac{\alpha}{\alpha+\beta}, 1, p+1\right) - \Phi\left(\frac{\alpha}{\alpha+\beta}, 2, p+1\right) \right\} \end{aligned} \quad (1)$$

$$\alpha H(p, \alpha, \beta) = p H(p-1, \alpha, \beta) - B(p-1, \alpha+\beta) + G(p-1, \alpha, \beta) \quad (1.1)$$

$$\int_0^\infty e^{-\alpha x} \ln x [-E_i(-\beta x)] dx = -\frac{1}{\alpha} \ln \left(1 + \frac{\alpha}{\beta}\right) [\gamma + \ln(\alpha + \beta)] - \frac{1}{(\alpha + \beta)} \Phi \left(\frac{\alpha}{\alpha + \beta}, 2, 1\right) \quad (1.2)$$

$$\begin{aligned} \int_0^\infty x e^{-\alpha x} \ln x [-E_i(-\beta x)] dx = & -\frac{1}{\alpha^2} \left[\ln \left(1 + \frac{\alpha}{\beta}\right) - 1 \left(\frac{\alpha}{\alpha + \beta}\right) \right] \\ & \times [\gamma + \ln(\alpha + \beta) - 1] - \frac{1}{(\alpha + \beta)^2} \Phi \left(\frac{\alpha}{\alpha + \beta}, 2, 2\right) \end{aligned} \quad (1.3)$$

$$\begin{aligned} \int_0^\infty x^2 e^{-\alpha x} \ln x [-E_i(-\beta x)] dx = & -\frac{2}{\alpha^3} \left[\ln \left(1 + \frac{\alpha}{\beta}\right) - 1 \left(\frac{\alpha}{\alpha + \beta}\right) - \frac{1}{2} \left(\frac{\alpha}{\alpha + \beta}\right)^2 \right] \\ & \times \left[\gamma + \ln(\alpha + \beta) - 1 - \frac{1}{2} \right] - \frac{2}{(\alpha + \beta)^3} \Phi \left(\frac{\alpha}{\alpha + \beta}, 2, 3\right) \end{aligned} \quad (1.4)$$

$$\begin{aligned} \int_0^\infty x^3 e^{-\alpha x} \ln x [-E_i(-\beta x)] dx = & -\frac{6}{\alpha^4} \left[\ln \left(1 + \frac{\alpha}{\beta}\right) - 1 \left(\frac{\alpha}{\alpha + \beta}\right) - \frac{1}{2} \left(\frac{\alpha}{\alpha + \beta}\right)^2 - \frac{1}{3} \left(\frac{\alpha}{\alpha + \beta}\right)^3 \right] \\ & \times \left[\gamma + \ln(\alpha + \beta) - 1 - \frac{1}{2} - \frac{1}{3} \right] - \frac{6}{(\alpha + \beta)^4} \Phi \left(\frac{\alpha}{\alpha + \beta}, 2, 4\right) \end{aligned} \quad (1.5)$$

$$\begin{aligned} \int_0^\infty x^n e^{-\alpha x} \ln x [-E_i(-\beta x)] dx = & -\frac{n!}{\alpha^{n+1}} \left[\ln \left(1 + \frac{\alpha}{\beta}\right) - \sum_{i=1}^n \frac{1}{i} \left(\frac{\alpha}{\alpha + \beta}\right)^i \right] \\ & \times \left[\gamma + \ln(\alpha + \beta) - \sum_{i=1}^n \frac{1}{i} \right] - \frac{n!}{(\alpha + \beta)^{n+1}} \Phi \left(\frac{\alpha}{\alpha + \beta}, 2, n+1\right) \end{aligned} \quad (1.6)$$

$$\int_0^\infty x^p \ln x [-E_i(-\beta x)] dx = \frac{\Gamma(p+1)}{(p+1)\beta^{p+1}} \left\{ [\psi(p+1) - \ln \beta] - \frac{1}{p+1} \right\} \quad (1.7)$$

$$z^n \Phi(z, 2, n+1) = \Phi(z, 2, 1) - \sum_{i=0}^{n-1} \frac{z^i}{(i+1)^2} \quad (1.8)$$

B.

$$\begin{aligned} H_1(p, \alpha, \beta, \rho) = \int_0^\rho x^p e^{-\alpha x} \ln x [-E_i(-\beta x)] dx = & \frac{\ln \rho}{\alpha^{p+1}} [-E_i(-\beta \rho)] \gamma(p+1, \alpha \rho) - \rho^{p+1} [-E_i(-\beta \rho)] \sum_{k=0}^\infty \frac{(-\alpha \rho)^k}{k! (p+k+1)^2} \\ & + \frac{1}{\beta^{p+1}} \left[\ln \rho \sum_{k=0}^\infty \frac{\left(-\frac{\alpha}{\beta}\right)^k}{k!} \frac{\gamma(p+k+1, \beta \rho)}{(p+k+1)} - \sum_{k=0}^\infty \frac{\left(-\frac{\alpha}{\beta}\right)^k}{k!} \frac{\gamma(p+k+1, \beta \rho)}{(p+k+1)^2} \right] \\ & - \rho^{p+1} \sum_{k=0}^\infty \frac{(-\alpha \rho)^k}{k! (p+k+1)} \sum_{l=0}^\infty \frac{(-\beta \rho)^l}{l! (p+k+l+1)^2} \end{aligned} \quad (2)$$

$$\alpha H_1(p, \alpha, \beta, \rho) = p H_1(p-1, \alpha, \beta, \rho) - B_1(p-1, \alpha + \beta, \rho) + G_1(p-1, \alpha, \beta, \rho) - \rho^p e^{-\alpha \rho} \ln \rho [-E_i(-\beta \rho)] \quad (2.1)$$

$$\int_0^\rho e^{-ax} \ln x [-E_i(-\beta x)] dx = -\frac{1}{\alpha} \ln \left(1 + \frac{\alpha}{\beta} \right) \left(\gamma + \ln(\alpha + \beta) + \left\{ -E_i[-(\alpha + \beta)\rho] \right\} \right)$$

Solution (a)

$$- \frac{1}{\alpha} \left[\left(\frac{\alpha}{\alpha + \beta} \right) \Phi \left(\frac{\alpha}{\alpha + \beta}, 2, 1 \right) + \ln \rho \left(e^{-a\rho} [-E_i(-\beta\rho)] - \left\{ -E_i[-(\alpha + \beta)\rho] \right\} \right) \right] \\ + \frac{1}{\alpha} \sum_{k=0}^{\infty} \frac{1}{k+1} \left(\frac{\alpha}{\alpha + \beta} \right)^{k+1} E_{k+2}[\rho(\alpha + \beta)]$$

$$= -\frac{1}{\alpha} \left\{ [\gamma + \ln(\alpha + \beta)] \ln \left(1 + \frac{\alpha}{\beta} \right) + \frac{\alpha}{\alpha + \beta} \Phi \left(\frac{\alpha}{\alpha + \beta}, 2, 1 \right) \right\} - \frac{1}{\alpha} [-E_i(-\beta\rho)]$$

Solution (b)

$$\times \left\{ \ln \rho e^{-a\rho} + [E_i(-\alpha\rho)] \right\} + \frac{1}{\alpha} \left(\frac{1}{2} \zeta(2) + \frac{1}{2} [\gamma + \ln(\alpha + \beta) + \ln \rho]^2 \right. \\ \left. + \ln \rho \left\{ -E_i[-(\alpha + \beta)\rho] \right\} + \sum_{k=0}^{\infty} \frac{[-(\alpha + \beta)\rho]^{k+1}}{k! (k+1)^3} \right) + \frac{1}{\alpha} \int_\rho^\infty \frac{e^{-\beta x} [-E_i(-\alpha x)]}{x} dx$$

$$= -\frac{1}{\alpha} \left\{ [\gamma + \ln(\alpha + \beta)] \ln \left(1 + \frac{\alpha}{\beta} \right) + \left(\frac{\alpha}{\alpha + \beta} \right) \Phi \left(\frac{\alpha}{\alpha + \beta}, 2, 1 \right) \right\} \\ + \frac{1}{\alpha} \left[(\gamma + \ln \beta \rho) [-E_i(-\alpha\rho)] + \frac{1}{2} \left\{ (\gamma + \ln \alpha \rho)^2 + [\gamma + \ln(\alpha + \beta)\rho]^2 + 2 \zeta(2) \right\} \right. \\ \left. + \sum_{k=0}^{\infty} \frac{[-\rho(\alpha + \beta)]^{k+1}}{k! (k+1)^3} + \sum_{k=0}^{\infty} \frac{(-\alpha\rho)^{k+1}}{k! (k+1)^3} + e^{-a\rho} \sum_{k=0}^{\infty} \frac{(-\beta/\alpha)^{k+1}}{(k+1)^2} e_k(\alpha\rho) \right. \\ \left. - \ln \rho \left(e^{-a\rho} [-E_i(-\beta\rho)] - \left\{ -E_i[-(\alpha + \beta)\rho] \right\} \right) \right]$$

Solution (c)

(good for $\alpha > \beta$)

$$= -\frac{1}{\alpha} \left\{ [\gamma + \ln(\alpha + \beta)] \ln \left(1 + \frac{\alpha}{\beta} \right) + \left(\frac{\alpha}{\alpha + \beta} \right) \Phi \left(\frac{\alpha}{\alpha + \beta}, 2, 1 \right) \right\} \\ + \frac{1}{\alpha} \left[- \left\{ \gamma + \ln \alpha \rho + [-E_i(-\alpha\rho)] \right\} [-E_i(-\beta\rho)] \right. \\ \left. + \frac{1}{2} \left\{ [\gamma + \ln \rho(\alpha + \beta)]^2 - (\gamma + \ln \beta \rho)^2 \right\} + \sum_{k=0}^{\infty} \frac{[-\rho(\alpha + \beta)]^{k+1}}{k! (k+1)^3} - \sum_{k=0}^{\infty} \frac{(-\rho\beta)^{k+1}}{k! (k+1)^3} - e^{-\beta\rho} \right. \\ \left. \times \sum_{k=0}^{\infty} \frac{(-\alpha/\beta)^{k+1}}{(k+1)^2} e_k(\beta\rho) - \ln \rho \left(e^{-a\rho} [-E_i(-\beta\rho)] - \left\{ -E_i[-(\alpha + \beta)\rho] \right\} \right) \right]$$

Solution (d)

(good for $\beta > \alpha$)

(2.2)

$$\int_0^p e^{-ax} \ln x [-E_i(-\alpha x)] dx = -\frac{1}{\alpha} \ln \rho \left\{ e^{-a\rho} [-E_i(-\alpha\rho)] - [-E_i(-2\alpha\rho)] \right\} \\ - \frac{1}{2\alpha} [-E_i(-\alpha\rho)]^2 + \frac{1}{2\alpha} (\gamma + \ln \alpha\rho)^2 + \frac{1}{\alpha} \sum_{k=0}^{\infty} \frac{(-2\alpha\rho)^{k+1}}{k! (k+1)^3} \quad (2.3)$$

$$\int_0^p x^p \ln x [-E_i(-\beta x)] dx = \frac{1}{\beta^{p+1} (p+1)!} \left(\ln \rho - \frac{1}{p+1} \right) \left\{ \gamma (p+1, \beta\rho) + (\beta\rho)^{p+1} [-E_i(-\beta\rho)] \right\} \\ - \frac{\rho^{p+1}}{(p+1)!} \sum_{k=0}^{\infty} \frac{(-\beta\rho)^k}{k! (p+k+1)^2} \quad (2.4)$$

$$\int_0^p x^n \ln x [-E_i(-\beta x)] dx = \frac{n!}{\beta^{n+1} (n+1)!} \left\{ [-E_i(-\beta\rho)] \left[\frac{\ln \rho (\beta\rho)^{n+1}}{n!} - \frac{(\beta\rho)^{n+1}}{(n+1)!} - 1 \right] \right. \\ \left. + e^{-\beta\rho} \left[\frac{e_n(\beta\rho)}{n+1} - \ln \rho e_n(\beta\rho) - \sum_{k=0}^{n-1} \frac{e_k(\beta\rho)}{k+1} \right] + \left[\psi(n+1) - \frac{1}{n+1} - \ln \beta \right] \right\} \quad (2.5)$$

$$z \Phi(z, 2, 1) = L_2(z) = - \int_0^z \frac{\ln(1-u) du}{u} \quad (\text{Euler's dilogarithm}) \quad (2.6)$$

$$L_2\left(\frac{1}{2}\right) = \frac{1}{2} [\zeta(2) - \ln^2(2)] \quad (2.7)$$

$$C. \quad H_2(p, \alpha, \beta, \rho) = \int_{\rho}^{\infty} x^p e^{-ax} \ln x [-E_i(-\beta x)] dx = H(p, \alpha, \beta) - H_1(p, \alpha, \beta, \rho) \quad (3)$$

$$\alpha H_2(p, \alpha, \beta, \rho) = p H_2(p-1, \alpha, \beta, \rho) - B_2(p-1, \alpha + \beta, \rho) + G_2(p-1, \alpha, \beta, \rho) + \rho^p e^{-a\rho} \ln \rho [-E_i(-\beta\rho)] \quad (3.1)$$

$$\int_{\rho}^{\infty} e^{-ax} \ln x [-E_i(-\beta x)] dx = \frac{1}{\alpha} \ln \rho \left(e^{-a\rho} [-E_i(-\beta\rho)] - \left\{ -E_i[-(\alpha + \beta)\rho] \right\} \right) \\ \text{Solution (a)} \quad + \frac{1}{\alpha} \ln \left(1 + \frac{\alpha}{\beta} \right) \left\{ -E_i[-(\alpha + \beta)\rho] \right\} - \frac{1}{\alpha} \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \left(\frac{\alpha}{\alpha + \beta} \right)^{k+1} E_{k+2}[\rho(\alpha + \beta)]$$

$$= \frac{1}{\alpha} [-E_i(-\beta\rho)] \left\{ \ln \rho e^{-a\rho} + [-E_i(-\alpha\rho)] \right\} - \frac{1}{\alpha} \left(\frac{1}{2} \zeta(2) + \frac{1}{2} [\gamma + \ln(\alpha + \beta) + \ln \rho]^2 \right) \\ \text{Solution (b)} \quad + \ln \rho \left\{ -E_i[-(\alpha + \beta)\rho] \right\} + \sum_{k=0}^{\infty} \frac{[-(\alpha + \beta)\rho]^{k+1}}{k! (k+1)^3} - \frac{1}{\alpha} \int_{\rho}^{\infty} \frac{e^{-\beta x} [-E_i(-\alpha x)]}{x} dx$$

Solution (c)
(good for $\alpha > \beta$)

$$\begin{aligned}
&= -\frac{1}{\alpha} \left[(\gamma + \ln \beta \rho) [-E_i(-\alpha \rho)] + \frac{1}{2} \left\{ (\gamma + \ln \alpha \rho)^2 + [\gamma + \ln(\alpha + \beta) \rho]^2 + 2 \zeta(2) \right\} \right. \\
&\quad + \sum_{k=0}^{\infty} \frac{[-\rho(\alpha + \beta)]^{k+1}}{k! (k+1)^3} + \sum_{k=0}^{\infty} \frac{(-\alpha \rho)^{k+1}}{k! (k+1)^3} + e^{-\alpha \rho} \sum_{k=0}^{\infty} \frac{(-\beta/\alpha)^{k+1}}{(k+1)^2} e_k(\alpha \rho) \\
&\quad \left. - \ln \rho \left(e^{-\alpha \rho} [-E_i(-\beta \rho)] - \left\{ -E_i[-(\alpha + \beta) \rho] \right\} \right) \right] \\
&= -\frac{1}{\alpha} \left[-\left\{ \gamma + \ln \alpha \rho + [-E_i(-\alpha \rho)] \right\} [-E_i(-\beta \rho)] + \frac{1}{2} \left\{ [\gamma + \ln(\alpha + \beta) + \ln \rho]^2 \right. \right. \\
&\quad \left. \left. - (\gamma + \ln \beta \rho)^2 \right\} + \sum_{k=0}^{\infty} \frac{[-\rho(\alpha + \beta)]^{k+1}}{k! (k+1)^3} - \sum_{k=0}^{\infty} \frac{(-\rho \beta)^{k+1}}{k! (k+1)^3} - e^{-\beta \rho} \sum_{k=0}^{\infty} \frac{(-\alpha/\beta)^{k+1}}{(k+1)^2} e_k(\beta \rho) \right. \\
&\quad \left. - \ln \rho \left(e^{-\alpha \rho} [-E_i(-\beta \rho)] - \left\{ -E_i[-(\alpha + \beta) \rho] \right\} \right) \right] \quad (3.2)
\end{aligned}$$

Solution (d)
(good for $\beta > \alpha$)

$$\begin{aligned}
\int_{\rho}^{\infty} e^{-\alpha x} \ln x [-E_i(-\alpha x)] dx &= \frac{1}{\alpha} \ln \rho \left\{ e^{-\alpha \rho} [-E_i(-\alpha \rho)] - [-E_i(-2\alpha \rho)] \right\} + \frac{1}{2\alpha} [-E_i(-\alpha \rho)]^2 \\
&\quad - \frac{1}{2\alpha} [(\gamma + \ln 2\alpha \rho)^2 + \zeta(2)] - \frac{1}{\alpha} \sum_{k=0}^{\infty} \frac{(-2\alpha \rho)^{k+1}}{k! (k+1)^3} \quad (3.3)
\end{aligned}$$

$$\begin{aligned}
\int_{\rho}^{\infty} x^p \ln x [-E_i(-\beta x)] dx &= \frac{\Gamma(p+1)}{(p+1)\beta^{p+1}} [\psi(p+1) - \ln \beta] - \frac{\Gamma(p+1, \beta \rho)}{(p+1)^2 \beta^{p+1}} - \frac{\ln \rho}{(p+1)\beta^{p+1}} \left\{ \gamma(p+1, \beta \rho) \right. \\
&\quad \left. + (\beta \rho)^{p+1} [-E_i(-\beta \rho)] \right\} + \frac{\rho^{p+1}}{(p+1)^2} [-E_i(-\beta \rho)] + \frac{\beta^{p+1}}{(p+1)} \sum_{k=0}^{\infty} \frac{(-\beta \rho)^k}{k! (p+k+1)^2} \quad (3.4)
\end{aligned}$$

VIII. INTEGRALS OF THE TYPE $\int x^p e^{-\alpha x} \ln^2 x [-E_i(-\beta x)] dx$

A.

$$\begin{aligned}
I(p, \alpha, \beta) &= \int_0^{\infty} x^p e^{-\alpha x} \ln^2 x [-E_i(-\beta x)] dx = \frac{\Gamma(p+1)}{(\alpha + \beta)^{p+1}} \left(\left\{ [\psi(p+1) - \ln(\alpha + \beta)]^2 + \zeta(2, p+1) \right\} \Phi\left(\frac{\alpha}{\alpha + \beta}, 1, p+1\right) \right. \\
&\quad \left. - 2 [\psi(p+1) - \ln(\alpha + \beta)] \Phi\left(\frac{\alpha}{\alpha + \beta}, 2, p+1\right) + 2 \Phi\left(\frac{\alpha}{\alpha + \beta}, 3, p+1\right) \right) \quad (1)
\end{aligned}$$

$$\alpha I(p, \alpha, \beta) = p I(p-1, \alpha, \beta) - C(p-1, \alpha+\beta) + 2H(p-1, \alpha, \beta) \quad (1.1)$$

$$\begin{aligned} \int_0^\infty e^{-\alpha x} \ln^2 x [-E_i(-\beta x)] dx &= \frac{1}{\alpha} \ln \left(1 + \frac{\alpha}{\beta} \right) \left\{ [\gamma + \ln(\alpha+\beta)]^2 + \zeta(2) \right\} \\ &+ \frac{2}{(\alpha+\beta)} [\gamma + \ln(\alpha+\beta)] \Phi \left(\frac{\alpha}{\alpha+\beta}, 2, 1 \right) + \frac{2}{(\alpha+\beta)} \Phi \left(\frac{\alpha}{\alpha+\beta}, 3, 1 \right) \end{aligned} \quad (1.2)$$

$$\begin{aligned} \int_0^\infty x e^{-\alpha x} \ln^2 x [-E_i(-\beta x)] dx &= \frac{1}{\alpha^2} \left[\ln \left(1 + \frac{\alpha}{\beta} \right) - 1 \left(\frac{\alpha}{\alpha+\beta} \right) \right] \left\{ [\gamma + \ln(\alpha+\beta) - 1]^2 + \zeta(2) - 1 \right\} \\ &+ \frac{2}{(\alpha+\beta)^2} [\gamma + \ln(\alpha+\beta) - 1] \Phi \left(\frac{\alpha}{\alpha+\beta}, 2, 2 \right) + \frac{2}{(\alpha+\beta)^2} \Phi \left(\frac{\alpha}{\alpha+\beta}, 3, 2 \right) \end{aligned} \quad (1.3)$$

$$\begin{aligned} \int_0^\infty x^2 e^{-\alpha x} \ln^2 x [-E_i(-\beta x)] dx &= \frac{2}{\alpha^3} \left[\ln \left(1 + \frac{\alpha}{\beta} \right) - 1 \left(\frac{\alpha}{\alpha+\beta} \right) - \frac{1}{2} \left(\frac{\alpha}{\alpha+\beta} \right)^2 \right] \left\{ \left[\gamma + \ln(\alpha+\beta) - 1 - \frac{1}{2} \right]^2 \right. \\ &+ \zeta(2) - 1 - \frac{1}{2^2} \left. \right\} + \frac{4}{(\alpha+\beta)^3} \left[\gamma + \ln(\alpha+\beta) - 1 - \frac{1}{2} \right] \Phi \left(\frac{\alpha}{\alpha+\beta}, 2, 3 \right) \\ &+ \frac{4}{(\alpha+\beta)^3} \Phi \left(\frac{\alpha}{\alpha+\beta}, 3, 3 \right) \end{aligned} \quad (1.4)$$

$$\begin{aligned} \int_0^\infty x^3 e^{-\alpha x} \ln^2 x [-E_i(-\beta x)] dx &= \frac{6}{\alpha^4} \left[\ln \left(1 + \frac{\alpha}{\beta} \right) - 1 \left(\frac{\alpha}{\alpha+\beta} \right) - \frac{1}{2} \left(\frac{\alpha}{\alpha+\beta} \right)^2 - \frac{1}{3} \left(\frac{\alpha}{\alpha+\beta} \right)^3 \right] \\ &\times \left\{ \left[\gamma + \ln(\alpha+\beta) - 1 - \frac{1}{2} - \frac{1}{3} \right]^2 + \zeta(2) - 1 - \frac{1}{2^2} - \frac{1}{3^2} \right\} \\ &+ \frac{12}{(\alpha+\beta)^4} \left[\gamma + \ln(\alpha+\beta) - 1 - \frac{1}{2} - \frac{1}{3} \right] \Phi \left(\frac{\alpha}{\alpha+\beta}, 2, 4 \right) + \frac{12}{(\alpha+\beta)^4} \Phi \left(\frac{\alpha}{\alpha+\beta}, 3, 4 \right) \end{aligned} \quad (1.5)$$

$$\begin{aligned} \int_0^\infty x^n e^{-\alpha x} \ln^2 x [-E_i(-\beta x)] dx &= \frac{n!}{\alpha^{n+1}} \left[\ln \left(1 + \frac{\alpha}{\beta} \right) - \sum_{k=0}^{n-1} \frac{1}{k+1} \left(\frac{\alpha}{\alpha+\beta} \right)^{k+1} \right] \left\{ \left[\gamma + \ln(\alpha+\beta) - \sum_{k=0}^{n-1} \frac{1}{k+1} \right]^2 + \zeta(2) \right. \\ &- \sum_{k=0}^{n-1} \frac{1}{(k+1)^2} \left. \right\} + \frac{2(n!)}{(\alpha+\beta)^{n+1}} \left[\gamma + \ln(\alpha+\beta) - \sum_{k=0}^{n-1} \frac{1}{k+1} \right] \Phi \left(\frac{\alpha}{\alpha+\beta}, 2, n+1 \right) \\ &+ \frac{2(n!)}{(\alpha+\beta)^{n+1}} \Phi \left(\frac{\alpha}{\alpha+\beta}, 3, n+1 \right) \end{aligned} \quad (1.6)$$

$$\int_0^\infty x^p \ln^2 x [-E_i(-\beta x)] dx = \frac{\Gamma(p+1)}{\beta^{p+1}(p+1)} \left(\left\{ [\psi(p+1) - \ln \beta]^2 + \zeta(2, p+1) \right\} - \frac{2}{p+1} [\psi(p+1) - \ln \beta] + \frac{2}{(p+1)^2} \right) \quad (1.7)$$

$$z^n \Phi(z, 3, n+1) = \Phi(z, 3, 1) - \sum_{k=0}^{n-1} \frac{z^k}{(k+1)^3} \quad (1.8)$$

B.

$$\begin{aligned}
I_1(p, \alpha, \beta, \rho) &= \int_0^\rho x^p e^{-\alpha x} \ln^2 x [-E_i(-\beta x)] dx = [-E_i(-\beta \rho)] \left[\frac{\gamma(p+1, \alpha \rho) \ln^2 \rho}{\alpha^{p+1}} - 2\rho^{p+1} \ln \rho \sum_{k=0}^{\infty} \frac{(-\alpha \rho)^k}{k! (p+k+1)^2} \right. \\
&\quad \left. + 2\rho^{p+1} \sum_{k=0}^{\infty} \frac{(-\alpha \rho)^k}{k! (p+k+1)^3} \right] + \frac{1}{\beta^{p+1}} \left[\ln^2 \rho \sum_{k=0}^{\infty} \frac{(-\alpha/\beta)^k}{k!} \frac{\gamma(p+k+1, \beta \rho)}{(p+k+1)} - 2 \ln \rho \sum_{k=0}^{\infty} \frac{(-\alpha/\beta)^k}{k!} \frac{\gamma(p+k+1, \beta \rho)}{(p+k+1)^2} \right. \\
&\quad \left. + 2 \sum_{k=0}^{\infty} \frac{(-\alpha/\beta)^k}{k!} \frac{\gamma(p+k+1, \beta \rho)}{(p+k+1)^3} \right] + 2\rho^{p+1} \left[\sum_{k=0}^{\infty} \frac{(-\alpha \rho)^k}{k! (p+k+1)^2} \sum_{l=0}^{\infty} \frac{(-\beta \rho)^l}{l! (p+k+l+1)^2} \right. \\
&\quad \left. + \sum_{k=0}^{\infty} \frac{(-\alpha \rho)^k}{k! (p+k+1)} \sum_{l=0}^{\infty} \frac{(-\beta \rho)^l}{l! (p+k+l+1)^3} - \ln \rho \sum_{k=0}^{\infty} \frac{(-\alpha \rho)^k}{k! (p+k+1)} \sum_{l=0}^{\infty} \frac{(-\beta \rho)^l}{l! (p+k+l+1)^2} \right] \quad (2)
\end{aligned}$$

$$\alpha I_1(p, \alpha, \beta, \rho) = p I_1(p-1, \alpha, \beta, \rho) - C_1(p-1, \alpha + \beta, \rho) + 2H_1(p-1, \alpha, \beta, \rho) - \rho^p e^{-\alpha \rho} \ln^2 \rho [-E_i(-\beta \rho)] \quad (2.1)$$

$$\begin{aligned}
\int_0^\rho e^{-\alpha x} \ln^2 x [-E_i(-\beta x)] dx &= \frac{1}{\alpha} [-E_i(-\beta \rho)] \left\{ \ln^2 \rho (1 - e^{-\alpha \rho}) - 2 \ln \rho \left\{ \gamma + \ln \alpha + \ln \rho + [-E_i(-\alpha \rho)] \right\} \right. \\
&\quad \left. - 2 \sum_{k=0}^{\infty} \frac{(-\alpha \rho)^{k+1}}{k! (k+1)^3} \right\} + \frac{1}{\beta} \left[\ln^2 \rho \sum_{k=0}^{\infty} \frac{(-\alpha/\beta)^k}{k! (k+1)} \gamma(k+1, \beta \rho) - 2 \ln \rho \sum_{k=0}^{\infty} \frac{(-\alpha/\beta)^k}{k! (k+1)^2} \right. \\
&\quad \times \gamma(k+1, \beta \rho) + 2 \sum_{k=0}^{\infty} \frac{(-\alpha/\beta)^k}{k! (k+1)^3} \gamma(k+1, \beta \rho) \left. \right] + 2\rho \left[\sum_{k=0}^{\infty} \frac{(-\alpha \rho)^k}{k! (k+1)^2} \sum_{l=0}^{\infty} \frac{(-\beta \rho)^l}{l! (k+l+1)^2} \right. \\
&\quad \left. + \sum_{k=0}^{\infty} \frac{(-\alpha \rho)^k}{k! (k+1)} \sum_{l=0}^{\infty} \frac{(-\beta \rho)^l}{l! (k+l+1)^3} - \ln \rho \sum_{k=0}^{\infty} \frac{(-\alpha \rho)^k}{k! (k+1)} \sum_{l=0}^{\infty} \frac{(-\beta \rho)^l}{l! (k+l+1)^2} \right] \quad (2.2)
\end{aligned}$$

$$\begin{aligned}
\int_0^\rho x^p \ln^2 x [-E_i(-\beta x)] dx &= \frac{1}{\beta^{p+1}(p+1)} \left[\ln^2 \rho - \frac{2 \ln \rho}{(p+1)} + \frac{2}{(p+1)^2} \right] \left\{ \gamma(p+1, \beta \rho) + (\beta \rho)^{p+1} [-E_i(-\beta \rho)] \right\} \\
&\quad - \frac{2\rho^{p+1}}{(p+1)} \left(\ln \rho - \frac{1}{p+1} \right) \sum_{k=0}^{\infty} \frac{(-\beta \rho)^k}{k! (p+k+1)^2} + \frac{2\rho^{p+1}}{(p+1)} \sum_{k=0}^{\infty} \frac{(-\beta \rho)^k}{k! (p+k+1)^3} \quad (2.3)
\end{aligned}$$

$$C. \quad I_2(p, \alpha, \beta, \rho) = \int_\rho^\infty x^p e^{-\alpha x} \ln^2 x [-E_i(-\beta x)] dx = I(p, \alpha, \beta) - I_1(p, \alpha, \beta, \rho) \quad (3)$$

$$\alpha I_2(p, \alpha, \beta, \rho) = p I_2(p-1, \alpha, \beta, \rho) - C_2(p-1, \alpha + \beta, \rho) + 2H_2(p-1, \alpha, \beta, \rho) + \rho^p e^{-\alpha \rho} \ln^2 \rho [-E_i(-\beta \rho)] \quad (3.1)$$

$$\int_\rho^\infty e^{-\alpha x} \ln^2 x [-E_i(-\beta x)] dx = \quad (3.2)$$

(See Sec. VIII, Eq. 1.2 and 2.2.)

$$\int_\rho^\infty x^p \ln^2 x [-E_i(-\beta x)] dx = \quad (3.3)$$

(See Sec. VIII, Eq. 1.7 and 2.3.)

IX. INTEGRALS OF THE TYPE $\int x^p e^{-ax} \ln^3 x [-E_i(-\beta x)] dx$

A.

$$J(p, \alpha, \beta) = \int_0^\infty x^p e^{-ax} \ln^3 x [-E_i(-\beta x)] dx = \frac{\Gamma(p+1)}{(\alpha+\beta)^{p+1}} \left(\left\{ [\psi(p+1) - \ln(\alpha+\beta)]^3 + 3\zeta(2, p+1) [\psi(p+1) - \ln(\alpha+\beta)] - 2\zeta(3, p+1) \right\} \Phi\left(\frac{\alpha}{\alpha+\beta}, 1, p+1\right) - 3 \left\{ [\psi(p+1) - \ln(\alpha+\beta)]^2 + \zeta(2, p+1) \right\} \times \Phi\left(\frac{\alpha}{\alpha+\beta}, 2, p+1\right) + 6 [\psi(p+1) - \ln(\alpha+\beta)] \Phi\left(\frac{\alpha}{\alpha+\beta}, 3, p+1\right) - 6 \Phi\left(\frac{\alpha}{\alpha+\beta}, 4, p+1\right) \right) \quad (1)$$

$$\alpha J(p, \alpha, \beta) = pJ(p-1, \alpha, \beta) - D(p-1, \alpha+\beta) + 6I(p-1, \alpha, \beta) \quad (1.1)$$

$$\begin{aligned} \int_0^\infty e^{-ax} \ln^3 x [-E_i(-\beta x)] dx = & -\frac{1}{\alpha} \ln\left(1 + \frac{\alpha}{\beta}\right) \left\{ [\gamma + \ln(\alpha+\beta)]^3 + 3\zeta(2) [\gamma + \ln(\alpha+\beta)] + 2\zeta(3) \right\} \\ & - \frac{3}{(\alpha+\beta)} \left\{ [\gamma + \ln(\alpha+\beta)]^2 + \zeta(2) \right\} \Phi\left(\frac{\alpha}{\alpha+\beta}, 2, 1\right) - \frac{6}{(\alpha+\beta)} [\gamma + \ln(\alpha+\beta)] \\ & \times \Phi\left(\frac{\alpha}{\alpha+\beta}, 3, 1\right) - \frac{6}{(\alpha+\beta)} \Phi\left(\frac{\alpha}{\alpha+\beta}, 4, 1\right) \end{aligned} \quad (1.2)$$

$$\begin{aligned} \int_0^\infty x^p \ln^3 x [-E_i(-\beta x)] dx = & \frac{\Gamma(p+1)}{(\alpha+\beta)^{p+1} (p+1)} \left(\left\{ [\psi(p+1) - \ln\beta]^3 + 3\zeta(2, p+1) [\psi(p+1) - \ln\beta] - 2\zeta(3, p+1) \right\} \right. \\ & \left. - \frac{3}{(p+1)} \left\{ [\psi(p+1) - \ln\beta]^2 + \zeta(2, p+1) \right\} + \frac{6}{(p+1)^2} [\psi(p+1) - \ln\beta] - \frac{6}{(p+1)^3} \right) \end{aligned} \quad (1.3)$$

B.

$$J_1(p, \alpha, \beta, \rho) = \int_0^\rho x^p e^{-ax} \ln^3 x [-E_i(-\beta x)] dx \quad (2)$$

$$\alpha J_1(p, \alpha, \beta, \rho) = pJ_1(p-1, \alpha, \beta, \rho) - D_1(p-1, \alpha+\beta, \rho) + 6I_1(p-1, \alpha, \beta, \rho) - \rho^p e^{-a\rho} \ln^3 \rho [-E_i(-\beta\rho)] \quad (2.1)$$

C.

$$J_2(p, \alpha, \beta, \rho) = \int_\rho^\infty x^p e^{-ax} \ln^3 x [-E_i(-\beta x)] dx \quad (3)$$

$$\alpha J_2(p, \alpha, \beta, \rho) = pJ_2(p-1, \alpha, \beta, \rho) - D_2(p-1, \alpha+\beta, \rho) + 6I_2(p-1, \alpha, \beta, \rho) + \rho^p e^{-a\rho} \ln^3 \rho [-E_i(-\beta\rho)] \quad (3.1)$$

X. MISCELLANEOUS RELATED INTEGRALS

A.

$$K(n, \alpha, \beta, p) = \int_0^\infty x^p e^{-\alpha x} (\ln x)^n [-E_i(-\beta x)] dx = \sum_{k=0}^n (-1)^k k! \binom{n}{k} \Phi\left(\frac{\alpha}{\alpha+\beta}, k+1, p+1\right) F(n-k, \alpha+\beta, \rho) \quad (1)$$

(See Sec. V, Eq. 1.)

$$\alpha K(n, \alpha, \beta, p) = pK(n, \alpha, \beta, p-1) - F(n, \alpha+\beta, p-1) + nK(n-1, \alpha, \beta, p-1) \quad (1.1)$$

B.

$$\alpha \int_\rho^\infty e^{-\alpha x} (\ln x)^n [-E_i(-\beta x)] dx = n \int_\rho^\infty \frac{e^{-\alpha x} (\ln x)^{n-1} [-E_i(-\beta x)] dx}{x} - \int_\rho^\infty \frac{e^{-(\alpha+\beta)x} (\ln x)^n dx}{x} + e^{-\alpha\rho} (\ln \rho)^n [-E_i(-\beta\rho)] \quad (2)$$

C.

$$\int_\rho^\infty \frac{e^{-\alpha x} [-E_i(-\beta x)] dx}{x} = \frac{1}{2} [(\gamma + \ln \beta\rho)^2 + \zeta(2)] + \sum_{k=0}^\infty \frac{(-\beta\rho)^{k+1}}{k!(k+1)^3} + [-E_i(-\beta\rho)] \left\{ \gamma + \ln \alpha\rho + [-E_i(-\alpha\rho)] \right\} + e^{-\beta\rho} \sum_{k=0}^\infty \frac{(-\alpha/\beta)^{k+1}}{(k+1)^2} e_k(\beta\rho) \quad (3)$$

$$\int_\rho^\infty \frac{e^{-\alpha x} [-E_i(-\beta x)] dx}{x} = [-E_i(-\alpha\rho)] [-E_i(-\beta\rho)] - \int_\rho^\infty \frac{e^{-\beta x} [-E_i(-\alpha x)] dx}{x} \quad (3.1)$$

$$\int_\rho^\infty \frac{[-E_i(-\alpha x)] dx}{x} = \int_\rho^\infty \frac{\ln x e^{-\alpha x} dx}{x} - \ln \rho [-E_i(-\alpha\rho)] \quad (3.2)$$

(See Sec. V, Eq. 3.4.)

$$\int_\rho^\infty \frac{[-E_i(-\alpha x)] dx}{x} = \frac{1}{2} [(\gamma + \ln \alpha\rho)^2 + \zeta(2)] + \sum_{k=0}^\infty \frac{(-\alpha\rho)^{k+1}}{k!(k+1)^3} \quad (3.3)$$

(See Sec. V, Eq. 3.5.)

$$\int_\rho^\infty \frac{e^{-\alpha x} [-E_i(-\alpha x)] dx}{x} = \frac{1}{2} [-E_i(-\alpha\rho)]^2 \quad (3.4)$$

D.

$$K_1(n, \alpha, \beta, p, \rho) = \int_0^\rho x^p e^{-\alpha x} (\ln x)^n [-E_i(-\beta x)] dx \quad (4)$$

$$\alpha K_1(n, \alpha, \beta, p, \rho) = pK_1(n, \alpha, \beta, p-1, \rho) - F_1(n, \alpha+\beta, p-1, \rho) + nK_1(n-1, \alpha, \beta, p-1, \rho) - \rho^p e^{-\alpha\rho} (\ln \rho)^n [-E_i(-\beta\rho)] \quad (4.1)$$

NOMENCLATURE

$\binom{a}{b}$	Binomial coefficients	$\frac{a!}{b!(a-b)!}$
$B_x(p,q)$	Incomplete Beta function	$\int_0^x t^{p-1}(1-t)^{q-1} dt$
$e_n(x)$	Truncated exponential	$\sum_{k=0}^n \frac{x^k}{k!}$
$[-E_i(-x)]$	Exponential integral	$\int_x^\infty e^{-t} t^{-1} dt$ $E_1(x)$
$E_n(x)$	Placzek function	$\int_1^\infty e^{-x't} t^{-n} dt$ $x^{n-1} \Gamma(1-n, x)$
${}_2F_1(a,b;c;z)$	Hypergeometric series	$\sum_{n=0}^\infty \frac{(a)_n (b)_n z^n}{(c)_n n!}$
$L_2(z)$	Euler's dilogarithm	$z \Phi(z, 2, 1)$ $\sum_{n=0}^\infty \frac{z^{n+1}}{(n+1)^2}$
$(p)_n$		$\frac{\Gamma(p+n)}{\Gamma(p)}$
γ	Euler's constant	.577 215 6649
$\gamma(a,x)$	Incomplete Gamma function	$\int_0^x e^{-t} t^{a-1} dt$
$\Gamma(a,x)$	Incomplete Gamma function	$\int_x^\infty e^{-t} t^{a-1} dt$
$\Gamma(z)$	Gamma function	$\int_0^\infty e^{-t} t^{z-1} dt$
$\zeta(s)$	Riemann Zeta function	$\zeta(s, 1)$ $\Phi(1, s, 1)$ $\sum_{n=0}^\infty \frac{1}{(n+1)^s}$
$\zeta(s,v)$	Generalized Zeta function	$\sum_{n=0}^\infty \frac{1}{(n+v)^s}$
$\Phi(z,s,v)$		$\sum_{n=0}^\infty \frac{z^n}{(n+v)^s}$
$\psi(z)$	Psi function	$\frac{d \ln \Gamma(z)}{dz}$

ACKNOWLEDGMENT

The author would like to express his gratitude to Dr. Robert G. Parr of the Department of Chemistry, Johns Hopkins University for his generous guidance and support during the course of this project.

REFERENCES

1. Erdelyi, A., W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions*, Vol. I, II, McGraw-Hill Book Company, Inc., New York, 1953.
2. Bierens de Haan, D., *Nouvelles tables d'intégrales définies*, Amsterdam, 1867.
3. Grobner, W., and N. Hofreiter, *Integraltafel*, Wien, Springer-Verlag, 1961.
4. Ryshik, I. M., and I. S. Gradstein, *Summen, Produkt- und Integral-Tafeln*, Veb Deutscher Verlag der Wissenschaften, Berlin, 1957.
5. LeCaine, J., *A Table of Integrals Involving the Functions $E_n(x)$* , Chalk River, Ontario, N.R.C., 1948.
6. Nielsen, N., *Handbuch der Theorie der Gamma Funktion*, B. G. Teubner, Leipzig, 1906.
7. Levenson, M., *American Mathematics Monthly*, Vol. 65, p. 695, 1958.
8. Busbridge, I. W., *Quarterly Journal of Mathematics*, Oxford Series (2), vol. 1, p. 176, 1950.
9. Kourganoff, K., *Annales d'Astrophysique*, Vol. 10, pp. 282, 329, 1947.